# The geometry of finite order jets of submanifolds and the variational formalism

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**Abstract**

We study the geometry of jets of submanifolds with special interest in the relationship with the calculus of variations. We give a new proof of the fact that higher order jets of submanifolds are affine bundles; as a by-product we obtain a new expression for the associated vector bundles. We use Green–Vinogradov formula to provide coordinate expressions for all variational forms, *i.e.*, objects in the finite-order variational sequence on jets of submanifolds. Finally, we formulate the variational problem in the framework of jets of submanifolds by an intrinsic geometric language, and connect it with the variational sequence. Detailed comparison with literature is provided throughout the paper.

**Keywords**: jets of submanifolds, contact elements, variational bicomplex, variational sequence, Vinogradov's $\mathcal{C}$-spectral sequence.

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## Contents

# Introduction

The study of jet spaces was born as the study of the notion of *contact*. From the analytical viewpoint, $r$-th contact between smooth maps was realized as the equality of their $r$-th differentials. From the geometrical viewpoint, $r$-th contact between manifolds was realised as the $r$-th contact of two local parametrisations of the given manifolds at a point. Clearly, the bridge from one viewpoint to another is provided by the notion of 'graph': two maps have an $r$-th contact if and only if their graphs have an $r$-th order contact.

It turns out that the most general notion of contact between manifolds can be given as contact between embedded submanifolds of a given manifold. A detailed justification of this statement is given in the subsection 1.2.

In the modern terminology, the spaces of equivalence classes of submanifolds of a given manifold having an $r$-th contact at a certain point are said to be *spaces of jets of submanifolds* (also known as *manifolds of contact elements*). The above arguments show that the geometry of jets of submanifolds plays a key role whenever dealing with mathematical topics where the geometry of contact is important, like geometric aspects of differential equations and calculus of variations. Despite this fact, the preferred approach to jets has been mostly through jets of maps or fibrings (see, *e.g.*, [7, 22, 30, 31, 39]), because the last ones generalise the notion of graph through the concept of 'section'. Instead, it is clear that jets of submanifolds are locally diffeomorphic to jets of fibrings, so jets of submanifolds constitute a non-trivial generalisation of jets of fibrings.

Historically, less attention has been devoted to jets of submanifolds with respect to other theories of jets. After the work of Lie, the research by Bompiani[1] was devoted to the geometry of jets of surfaces about 1910 and during all his career [5]. He was especially concerned in projective invariants of jets of curves and surfaces and applications to differential equations. It seems that his approach has been left apart. He also wrote a remarkable work on the history of the notion of contact, which he dates back to Ruffini [6].

Jets of submanifolds have been first introduced in a modern setting by C. Ehresmann [14]. He defined the notion of contact of order $r$ between parametrised submanifolds, then identified $r$-equivalent parametrisations. The works [8, 9, 10, 19, 20, 22, 24, 29, 35], are among the few researches performed within this scheme. We also mention

[1]He used the name *elementi differenziabili* to indicate jets of submanifolds

the brief introduction in [37] (under the name of *extended jet bundles*). In the classical calculus of variations, jets of submanifolds arise implicitly when considering parametric problems [15][2]. In such variational problems the Lagrangian is invariant with respect to changes of parametrisation, hence it factors to the space of jets of submanifolds with respect to the action of change of parametrisation.

More directly, in [11, 43, 44, 45] jets of submanifolds are introduced through the notion of contact of order $r$ between submanifolds. In this paper, we followed this second approach. Indeed, we find this viewpoint more straightforward, especially for what concerns coordinate expressions, because the dependency on parametrisation is factored out. Authors working in the parametrised scheme like [17, 18, 19, 8, 9, 10] are forced to check each time the independence of their computations from parametrisations.

Our guideline in the study of jets of submanifolds has been to following: *to proceed by analogy with other theories of jets*, in particular jets of fibrings, and try to reproduce similar geometric structures and constructions.

This is our second paper about jets of submanifolds. In the previous one [33] we introduced the pseudo-horizontal and pseudo-vertical bundles on jets of submanifolds (see subsection 1.3). Such bundles play the role of horizontal and vertical bundles in the case of jets of fibrings. By those bundles we computed the cohomology of the finite order analogue of Vinogradov's $\mathcal{C}$-spectral sequence (see section 2).

In this paper we begin with a more careful exposition of basic definitions (subsections 1.1, 1.3), together with a thorough comparison between the three known approaches to jet spaces (subsection 1.2). Then, we derive a new proof of the fact that higher-order jets of submanifolds are affine bundles (subsection 1.4). Other proofs in literature [4, 24, 35] rely on the analysis of the transformation group of the fibres. Our proof is based on the structure of the pseudo-horizontal and pseudo-vertical bundles, and is a generalisation of the similar result in the case of jets of fibrings. As a by-product, we are able to provide a new explicit expression of the associated vector bundles using tensor products of pseudo-horizontal and pseudo-vertical bundles.

Next, we recall briefly the computations that led in [33] to the computation of the finite order variational sequence on jets of submanifolds. The sequence contains spaces of forms with a special polynomial structure. Such a structure has been studied in coordinates so far (*hyperjacobians*, [36], and [2], [3, chap. 4]). In this paper we find it as a by-product of our geometric structure (subsection 1.5 and section 2). We also provide a useful coordinate expression for the first differential of the spectral sequence, $\bar{d}$ (proposition 2).

Then, we show that a finite order version of Green-Vinogradov formula does not exist (subsection 3.1). This leads us to use the infinite-order theory to compute the representative of each cohomology class (also known as *variational form*) in the variational sequence. Such representatives are computed in theorem 2 through (22). As a by-product, we solve a problem from [16] (conjecture/question by P. Griffiths, remark 7). This is due to the fact that we do not confine ourselves to the computation of the horizontal cohomology (*i.e.*, the cohomology of the horizontal de Rham

[2] Such problems date back to Carathéodory in the case of one independent variable

sequence [7, 25, 26, 42]), but we compute all cohomology groups of the finite order $\mathcal{C}$-spectral sequence.

We provide the coordinate expression for all representatives of variational forms in subsection 3.2. In the cases $p = 0, 1, 2$ these expressions look like the well-known expressions of Lagrangians, Euler–Lagrange operators and Helmholtz operators of the case of jets of fibrings, but the spaces where they are defined are quite different, hence also their coordinate transformation laws. As far as we know, it is the first time that such expressions are computed on jets of submanifolds.

Finally, in section 4 we provide a geometric (*i.e.*, invariant, coordinate-free) formulation of the variational problem on jets of submanifolds. This allows us to derive the Euler–Lagrange equations in an intrinsic way. Our formulation reduces to well-known formulations in the case of jets of fibrings (see [22, 29, 30, 39, 41], for example), and is a radical improvement of the old formulation by Dedecker [11]. We also describe the connection of this formalism with the finite order $\mathcal{C}$-spectral sequence. Indeed, we think that this formalism provides one of the main motivations for the $\mathcal{C}$-spectral sequence itself.

We already observed that variational problems on jets of submanifolds have been introduced in the parametric framework [8, 9, 10, 17, 18]. In our opinion, we provided a simpler formulation through pseudo-horizontal bundles. As an example, we deal with a single Lagrangian, not with a set of Lepage equivalents. Moreover, we do not need to check the invariance with respect to changes of parametrisations. We added detailed comparisons with literature in section 4 explaining our viewpoint.

As a final remark, we regret that we had no time and space to support our theory with examples, like in [8]. We have some partial results [32], that we hope to work out and expose in a next paper.

# 1 Jet spaces

In this section we recall basic facts about the geometry of jets of submanifolds (our sources were [4, 34, 46]) together with new structures (subsections 1.3, 1.4) which will be used for the analysis of the finite order $\mathcal{C}$-spectral sequence.

In this paper all manifolds and maps are smooth, and all submanifolds are *embedded submanifolds*.

We shall introduce jets through the fundamental notion of contact between maps. This is sometimes given through composition of maps with curves (see, *e.g.*, [22]); here, we adopt a more direct viewpoint.

Let $N$ and $M$ be two manifolds, and $f, g\colon N \longrightarrow M$ two maps. We say that $f$ and $g$ have a *contact of order* $r$ at $x$ if there is a chart $(U, (x^\lambda))$ at $x$ and a chart $(V, (u^i))$ at $f(x)$ such that

$$J^k((u^i) \circ f \circ (x^\lambda)^{-1}) = J^k((u^i) \circ g \circ (x^\lambda)^{-1}), \qquad 0 \leq k \leq r,$$

where $J^k$ stands for the $k$-th order differential. Of course, $f$ and $g$ have a contact of order zero if $f(x) = g(x)$.

It is easy to realize that the above notion is independent of the chosen chart.

## 1.1 Jet spaces

Let $E$ be an $(n+m)$-dimensional manifold.

Let $(V,(y^h))$ be a local chart on $E$. Any splitting $\{1,\dots,n+m\} = I_n \cup I_m$ into two subsets $I_n$ and $I_m$ having, respectively, $n$ and $m$ elements, induces a splitting $(y^h) = (x^\lambda, u^i)$ of the chart, where $x^\lambda = y^{h_\lambda}$, $u^i = y^{h_i}$ and $h_\lambda \in I_n$, $h_i \in I_m$, so that $1 \le \lambda \le n$ and $1 \le i \le m$. The above splitting is said to be a *division* of the chart $(y^h)$. A chart of the form $(x^\lambda, u^i)$ is said to be a *divided chart*. Of course, there are $\binom{n+m}{n}$ possible divisions of $(y^h)$.

In what follows, Greek indices run from 1 to $n$ and Latin indices run from 1 to $m$.

We say that a divided chart $(V,(x^\lambda,u^i))$ is *fibred* if $V$ is diffeomorphic to $X\times U$, where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are open subsets. Of course, the trivial projection $\pi\colon V \longrightarrow X$ makes $V$ a fibred manifold on $X$.

Let $L$ be an (embedded) submanifold $\iota\colon L \hookrightarrow E$ ($\iota$ is the inclusion map). We say that a divided chart $(V,(x^\lambda,u^i))$ is *concordant* with $L$ at $x \in V\cap L$ if the coordinate expression of $\iota$ is

$$(x^\lambda, u^i)\circ\iota = (x^\lambda, \iota^i),$$

where the functions $\iota^i$ are smooth functions of $(x^\lambda)$. We say that the chart is *adapted* to $L$ at $x$ if $(\iota^i) = (0)$. Such a chart exists at any $x \in L$.

Let $\iota\colon L \hookrightarrow E$, $\iota'\colon L' \hookrightarrow E$ be two submanifolds, and $x \in L\cap L'$. Let $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\dots,\sigma_r)$, with $1 \le \sigma_i \le n$ and $r \in \mathbb{N}$, be a multi-index, and $|\boldsymbol{\sigma}| \overset{\text{def}}{=} r$. Then, we say that $L$ and $L'$ have a *contact of order* $r$ at $x$ if $\iota$ and $\iota'$ have a contact of order $r$ at $x$.

Let us examine the above definition in coordinates. It can be proved that there exists a fibred chart which is concordant to both $L$ and $L'$ (by the theory of transversality, see, *e. g.*, [21]). If $(x^\lambda,u^i)$ is such a chart at $x$, then $L$ and $L'$ have a contact of order $r$ at $x \in L\cap L'$ if

$$\text{(1)}\qquad \frac{\partial^{|\boldsymbol{\sigma}|}\iota^i}{\partial x^{\sigma_1}\cdots\partial x^{\sigma_r}}(x) = \frac{\partial^{|\boldsymbol{\sigma}|}\iota'^i}{\partial x^{\sigma_1}\cdots\partial x^{\sigma_r}}(x), \qquad 0 \le |\boldsymbol{\sigma}| \le r.$$

The relation "contact of order $r$ between submanifolds at $x \in E$" is an equivalence relation; an equivalence class is denoted by $[L]^r_x$.

**Definition 1.** Let

$$J^r(E,n) \overset{\text{def}}{=} \bigcup_{x\in E} J^r_x(E,n),$$

where $J^r_x(E,n)$ is the set of the equivalence classes $[L]^r_x$ of $n$-dimensional submanifolds $L \subset E$ having a contact of order $r$ at $x$. We call $J^r(E,n)$ the *$r$-jet of $n$-dimensional submanifolds of $E$.*

Note that $J^0(E,n) = E$. Any submanifold $L \subset E$ can be prolonged to a subset of $J^r(E,n)$ through the injective map

$$j_r L \colon L \longrightarrow J^r(E,n), \quad x \longmapsto [L]^r_x.$$

The set $J^r(E,n)$ has a natural manifold structure: any divided chart $(V, x^\lambda, u^i)$ induces the local chart $\big(V^r_n, (x^\lambda, u^i_{\boldsymbol{\sigma}})\big)$ on $J^r(E,n)$ at $[L]^r_x$, where $V^r_n \overset{\text{def}}{=} \bigcup_{x\in V} J^r_x(E,n)$, and the functions $u^j_{\boldsymbol{\sigma}}$ are determined by

$$u^i_{\boldsymbol{\sigma}} \circ j_r L = \frac{\partial^{|\boldsymbol{\sigma}|} \iota^i}{\partial x^{\sigma_1} \cdots \partial x^{\sigma_r}}.$$

Coordinate changes are smooth because they involve compositions and sums of $k$-th order Jacobians $(0 \le k \le r)$ of the coordinate change in $E$. The dimension of $J^r(E,n)$ is the sum of the dimension of $E$ with the number of all the possible derivatives of $u^i$ with respect to $x^\lambda$ up to order $r$, hence $\dim J^r(E,n) = n + m \sum_{h=0}^r \binom{n+h-1}{h} = n + m\binom{n+r}{r}$.

The map $j_r L$ turns out to be an embedding; we shall identify it with its image, denoting it by $L^{(r)}$.

Any local diffeomorphism $F \colon E \longrightarrow E$ prolongs to a fibred isomorphism $J^r F \colon J^r(E,n) \longrightarrow J^r(E,n)$ in a way similar to jets of fibrings (see, for example, [31, 39]). Namely, we have $J^r F([L]^r_x) = [F(L)]^r_{F(x)}$. Hence, any vector field $X \colon E \longrightarrow TE$ prolongs to a vector field $X_r \colon J^r(E,n) \longrightarrow TJ^r(E,n)$, by prolonging its flow.

The above smooth manifold structure endows the natural projections

$$\pi_{r,h} : J^r(E,n) \longrightarrow J^h(E,n), \quad r \ge h,$$

with a bundle structure. In particular, it is known [4, 24, 45] that $\pi_{r+1,r}$ are affine bundles for $r \ge 1$. Later on, we shall provide a more complete proof in our framework. We shall also denote by $J^\infty(E,n)$ the inverse limit of the chain of projections

$$\cdots \xrightarrow{\pi_{r+1,r}} J^r(E,n) \xrightarrow{\pi_{r,r-1}} \cdots \xrightarrow{\pi_{2,1}} J^1(E,n) \xrightarrow{\pi_{1,0}} E.$$

A special attention is needed in the case $r = 0$. It is easy to realize that $\pi_{1,0}$ coincides with the Grassmann bundle of $n$-dimensional subspaces in $TE$. Any chart at $x \in E$ induces a covering of the Grassmann space $\mathrm{Gr}(T_x E, n) \simeq \pi^{-1}_{r,0}(x)$ $(x^\lambda, u^i)$ as follows. Any division $(x^\lambda, u^i)$ of the given chart induces a local chart of $\mathrm{Gr}(T_x E, n)$ onto the open set made by $n$-dimensional subspaces spanned by the vectors

$$\frac{\partial}{\partial x^\lambda}\bigg|_x + u^i_\lambda \frac{\partial}{\partial u^i}\bigg|_x . \tag{2}$$

Each of the above open subsets is dense in $\mathrm{Gr}(T_x E, n)$, and the set of all divisions of the given chart covers $\mathrm{Gr}(T_x E, n)$

Note that $\pi_{1,0}$ has no affine structure (this contrasts with the case of jets of fibrings, see [31, 39]). It is easy to realize that $\pi_{1,0}$ has a Grassmannian structure, in the sense that fibred coordinate changes are Grassmannian transformations induced by

isomorphisms of $TE$. More precisely, let $(x^\lambda, u^i)$ and $(y^\mu, v^j)$ be two coordinate charts concordant to the same submanifold $L \subset E$; let us denote by $(J^\mu_\lambda, J^\mu_i, J^j_\lambda, J^j_i)$ the Jacobian of the change of coordinates. Then the fibred coordinate change is given by the following formula

$$v^j_\mu = \frac{J^j_\lambda + J^j_i u^i_\lambda}{J^\mu_\lambda + J^\mu_i u^i_\lambda}. \tag{3}$$

In fact, locally, the submanifold $L$ is expressed by $u^i = f^i(x^\lambda)$ or $v^j = g^j(y^\mu)$. So, differentiating the equation $v^j|_L = v^j(x^\lambda, f^i(x^\lambda)) = g^j\left(y^\mu(x^\lambda, f^i(x^\lambda))\right)$ we get the result.

## 1.2 Jets of submanifolds as a generalisation of jets of fibrings and maps

We hereafter provide justifications to the title of this subsection, of intrinsic and local nature. Suppose that $E$ is endowed with a fibring $\pi : E \longrightarrow M$.

- The space $J^r\pi$ of $r$-th jets of sections $s\colon M \longrightarrow E$ of $\pi$ is an open dense subset of $J^r(E,n)$. In fact, it coincides with the subset of $r$-th jets of submanifolds of the type $s(M)$, and it is covered by just one of the open subsets of the previous covering of $\mathrm{Gr}(T_xE, n)$ (2). See [7, 26, 31, 39] for the theory of jets of fibrings.
- When $E = X \times U$, then there is the trivial fibring $\mathrm{pr}\colon X \times U \longrightarrow X$, and $J^r\,\mathrm{pr}$ coincides with the space of $r$-jets of maps $J^r(X, U)$. In fact, the space $J^r(X, U)$ is defined as the set of equivalence classes of maps of the type $f\colon X \longrightarrow U$ having an $r$-contact at a point, hence we fall back in the previous case. See [21, 22] for definitions and properties of jets of maps.
- A comparison between transformation rules of independent and dependent variables in various approaches to jets also shows that jets of submanifolds are the most general theory [11, 46]. Let $(x^\lambda, u^i)$ and $(\bar{x}^\lambda, \bar{u}^i)$ be two charts at the same point. Then we have the transformation rules:
  - $(x^\lambda(\bar{x}^\mu), u^i(\bar{u}^j))$ (jets of maps),
  - $(x^\lambda(\bar{x}^\mu), u^i(\bar{x}^\mu, \bar{u}^j))$ (jets of fibrings),
  - $(x^\lambda(\bar{x}^\mu, \bar{u}^j), u^i(\bar{x}^\mu, \bar{u}^j))$ (jets of submanifolds).

It should be said that $J^r(E,n)$ can be derived from jets of maps as follows. Consider the subspace $\mathrm{Imm}\, J^r_0(\mathbb{R}^n, E) \subset J^r(\mathbb{R}^n, E)$ of $r$-jets of immersions at $0 \in \mathbb{R}^n$. Any immersion can be seen as a local parametrisation of an (embedded) submanifold of $M$. There is an action of the group $G^r_n$ of $r$-jets of local diffeomorphisms $f$ of $\mathbb{R}^n$ such that $f(0) = 0$ on $\mathrm{Imm}\, J^r_0(\mathbb{R}^n, E)$; this can be regarded as the action by changes of parametrisation. Then it is easy to see that the following isomorphism holds

$$\mathrm{Imm}\, J^r_0(\mathbb{R}^n, E)/G^r_n \simeq J^r(E,n).$$

This can be taken as a definition of jets of submanifolds: see [17, 18, 19, 22, 29].

*Remark* 1. Some authors prefer to work on $\operatorname{Imm} J^r_0(\mathbb{R}^n, E)$ with objects that factor through the action by $G^r_n$ to $J^r(E,n)$ [17, 18, 19, 8, 9, 10]. This amounts to work with $n$ extra parameters and to show each time the independence of the parameters. In our opinion, this method seems to be more involved with respect to the direct use of jets of submanifolds. We especially get results in a more straightforward way with respect to the 'parametric' approach in section 4.

## 1.3 Contact sequence

Here, we introduce a new definition of contact structure on jets which turns out to be very useful in some cases, like in the analysis of finite order variational sequences [33]. This idea generalises the definition that is given in [34] for first-order jets, but it is somehow different from the standard contact (or Cartan) distribution on jets which is of fundamental importance for the geometric study of differential equations [7].

For $r \geq 0$ consider the following bundles over $J^{r+1}(E,n)$: the pull-back bundle

$$T^{r+1,r} \stackrel{\text{def}}{=} J^{r+1}(E,n) \underset{J^r(E,n)}{\times} TJ^r(E,n), \tag{4}$$

the subbundle $H^{r+1,r}$ of $T^{r+1,r}$ defined by

$$H^{r+1,r} \stackrel{\text{def}}{=} \left\{ \left([L]^{r+1}_p, v\right) \in T^{r+1,r} \mid v \in T_{[L]^r_p} L^{(r)} \right\} \tag{5}$$

, and the quotient bundle

$$V^{r+1,r} \stackrel{\text{def}}{=} T^{r+1,r}/H^{r+1,r}. \tag{6}$$

The bundles $H^{r+1,r}$ and $V^{r+1,r}$ are strictly related with the horizontal and vertical bundle in the case of jets of a fibring (see remark 2).

**Definition 2.** We call $H^{r+1,r}$ and $V^{r+1,r}$, respectively, the *pseudo-horizontal* and the *pseudo-vertical* bundle of $J^r(E,n)$.

First of all, we see how the above bundles relate to the projections $\pi_{r+1,r}$. There is a natural projection $T^{r+1,r} \longrightarrow T^{r,r-1}$. It restricts to $H^{r+1,r} \longrightarrow H^{r,r-1}$, hence it induces a natural projection $V^{r+1,r} \longrightarrow V^{r,r-1}$.

We observe that, while $(T^{r+1,r})^*$ is a vector subbundle of $T^*J^{r+1}(E,n)$ via the inclusion $T^*\pi_{r+1,r}$, there is no natural inclusion of $T^{r+1,r}$ into $TJ^{r+1}(E,n)$.

The pseudo-horizontal bundle has some additional features. The following isomorphism over $\mathrm{id}_{J^{r+1}(E,n)}$ holds

$$H^{r+1,r} \longrightarrow J^{r+1}(E,n) \times_{J^1(E,n)} H^{1,0}, \qquad ([L]^{r+1}_x, v) \longmapsto \left([L]^{r+1}_x, T\pi_{r,0}(v)\right). \tag{7}$$

Hence, we obtain the natural projection $(H^{r+1,r})^* \longrightarrow (H^{r,r-1})^*$.

Note that, if $L \subset E$ is an $n$-dimensional submanifold of $E$, then the pull-back bundle $(j_{r+1}L)^* H^{r+1,r}$ on $L$ is just $TL$, as it is easily seen. This is a further justification for its

name: indeed, the pseudo-horizontal bundle is tangent to the jet of any $n$-dimensional submanifold of $E$.

The most important property of the bundles $T^{r+1,r}$, $H^{r+1,r}$ and $V^{r+1,r}$ is the following *contact exact sequence*

$$0 \longrightarrow H^{r+1,r} \xhookrightarrow{D^{r+1}} T^{r+1,r} \xrightarrow{\omega^{r+1}} V^{r+1,r} \longrightarrow 0, \tag{8}$$

where $D^{r+1}$ and $\omega^{r+1}$ are the natural inclusion and quotient projection. It induces the following exact sequence

$$0 \longleftarrow \bigwedge^k (H^{r+1,r})^* \xleftarrow{\bigwedge^k (D^{r+1})^*} \bigwedge^k (T^{r+1,r})^* \xleftarrow{(\omega^{r+1})^* \wedge \mathrm{id}} (V^{r+1,r})^* \wedge \bigwedge^{k-1} (T^{r+1,r})^* \longleftarrow 0. \tag{9}$$

*Remark* 2. When $E$ is endowed with a fibring $\pi : E \longrightarrow M$, we have the following isomorphisms over $J^{r+1}\pi$:

$$H^{r+1,r}|_{J^{r+1}\pi} \longrightarrow J^{r+1}\pi \times_M TM, \qquad V^{r+1,r}|_{J^{r+1}\pi} \longrightarrow J^{r+1}\pi \times_{J^r\pi} VJ^r\pi,$$

where $VJ^r\pi \overset{\text{def}}{=} \ker T\pi_{r,0}$. Hence the exact sequence over $J^{r+1}\pi$

$$0 \longrightarrow J^{r+1}\pi \times_{J^r\pi} VJ^r\pi \hookrightarrow J^{r+1}\pi \times_{J^r\pi} TJ^r\pi \xrightarrow{T(\pi\circ\pi_{r,0})} J^{r+1}\pi \times_M TM \longrightarrow 0$$

splits the contact sequence (8) [31, 39].

Let us evaluate the coordinate expressions of $D^{r+1}$ and $\omega^{r+1}$. To this end, we observe that $D^{r+1}$ and $\omega^{r+1}$ can be seen as sections of the bundles $(H^{r+1,r})^* \otimes_{J^{r+1}(E,n)} T^{r+1,r}$ and $(T^{r+1,r})^* \otimes_{J^{r+1}(E,n)} V^{r+1,r}$ respectively.

A local basis of the space of sections of the bundle $H^{r+1,r}$ is

$$D^{r+1}_\lambda = \frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma},\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}},$$

where the index $\boldsymbol{\sigma},\lambda$ stands for $(\sigma_1,\dots,\sigma_s,\lambda)$ with $s \le r$. A local basis of the space of sections $(H^{r+1,r})^*$ dual to $(D^{r+1}_\lambda)$ is given by the restriction of the 1-forms $dx^\lambda$ to $H^{r+1,r}$, and is denoted by $\overline{dx}^\lambda$. The local expression of $D^{r+1}$ turns out to be

$$D^{r+1} = \overline{dx}^\lambda \otimes D^{r+1}_\lambda = \overline{dx}^\lambda \otimes \left( \frac{\partial}{\partial x^\lambda} + u^j_{\boldsymbol{\sigma},\lambda} \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}} \right).$$

A local basis of the space of sections of the bundle $V^{r+1,r}$ is

$$B^{\boldsymbol{\sigma}}_j \overset{\text{def}}{=} \left[ \frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}} \right], \qquad |\boldsymbol{\sigma}| \le r.$$

The local expression of $\omega^{r+1}$ is

$$\omega^{r+1} = \omega^j_{\boldsymbol{\sigma}} \otimes B^{\boldsymbol{\sigma}}_j = \left( du^j_{\boldsymbol{\sigma}} - u^j_{\boldsymbol{\sigma},\lambda} dx^\lambda \right) \otimes B^{\boldsymbol{\sigma}}_j.$$

*Remark* 3. We have a natural distribution $\mathcal{C}^r$ on $J^r(E,n)$ generated by the tangent spaces $TL^{(r)}$ for any $n$-dimensional submanifold $L \subset E$, namely the *Cartan distribution* [4, 7]. It is generated by the vector fields $D^r_\lambda$ and $\partial/\partial u^i_{\boldsymbol{\sigma}}$, with $|\boldsymbol{\sigma}| = r$. This distribution has not to be confused with $H^{r,r-1}$, which is a subbundle of a different bundle and is generated by $D^r_\lambda$.

## 1.4 Bundle structures

Here we give a proof of the fact that $(J^{r+1}(E,n), \pi_{r+1,r}, J^r(E,n))$ are affine bundles if $r \geq 1$. This is already known [4, 24, 35], but our proof is different and has the advantage to provide also a new expression of the associated vector bundle in terms of our pseudo-horizontal and pseudo-vertical bundles.

We stress that the proof is almost completely analogous to the case of jets of fibrings [31], having introduced the analogues of horizontal and vertical bundles in the previous subsection. The only difference is the absence of the base space in our case, giving the obstruction to $\pi_{1,0}$ to be affine.

**Lemma 1.** *The following isomorphism holds*

$$VJ^1(E,n) \simeq (H^{1,0})^* \otimes_{J^1(E,n)} V^{1,0}.$$

*Proof.* Any point of $J^1(E,n)$ can be seen as the inclusion of an $n$-dimensional subspace of $TE$ into $TE$ itself through $D^1$, hence as a linear map $\overline{dx}^\lambda \otimes (\partial/\partial x^\lambda + u^i_\lambda \partial/\partial u^i)$. A curve tangent to the fibre of $\pi_{1,0}$ at such a point has the tangent vector $\overline{dx}^\lambda \otimes \dot{u}^i_\lambda \partial/\partial u^i$; this proves the above isomorphism. □

**Theorem 1.** *For $r \geq 1$ the bundles $(J^{r+1}(E,n), \pi_{r+1,r}, J^r(E,n))$ are affine bundles associated with the vector bundle*

$$\left(\left(\odot^{r+1}(H^{1,0})^*\right) \otimes_{J^r(E,n)} V^{1,0}, \mathrm{pr}, J^r(E,n)\right),$$

*where* pr *is the trivial pull-back projection.*

*Proof.* We shall prove the theorem in two steps, dealing separately with the cases $r=1$ and $r>1$.

For $r=1$ we can interpret $D^1$ as the section

$$D^1 \colon J^1(E,n) \longrightarrow (H^{1,0})^* \otimes_{J^1(E,n)} TE.$$

For $r>1$ we can interpret $D^r$ as a fibred inclusion

$$D^r \colon J^r(E,n) \longrightarrow (H^{1,0})^* \otimes_{J^{r-1}(E,n)} TJ^{r-1}(E,n)$$

of bundles over $\mathrm{id}_{J^{r-1}(E,n)}$ through the isomorphism (7).

For $r=1$ we have the following commutative diagram

$$\begin{tikzcd} J^2(E,n) \arrow[r, "D^2"] \arrow[d, "\pi_{2,1}"'] & (H^{1,0})^* \otimes_{J^1(E,n)} TJ^1(E,n) \arrow[d, "\mathrm{id} \otimes T\pi_{1,0}"] \\ J^1(E,n) \arrow[r, "D^1"] & (H^{1,0})^* \otimes_{J^1(E,n)} TE \end{tikzcd}$$

The inverse image through $\mathrm{id} \otimes T\pi_{1,0}$ of the non-vanishing section $D^1(J^1(E,n))$ is an affine bundle. In view of lemma 1 the associated vector bundle is

$$\ker(\mathrm{id} \otimes T\pi_{1,0}) = (H^{1,0})^* \otimes_{J^1(E,n)} VJ^1(E,n) \simeq (H^{1,0})^* \otimes_{J^1(E,n)} (H^{1,0})^* \otimes_{J^1(E,n)} V^{1,0}.$$

The above affine bundle contains $J^2(E,n)$ as the subbundle $D^2(J^2(E,n))$. It is easy to realise that $J^2(E,n) \longrightarrow J^1(E,n)$ is the affine subbundle whose associated vector bundle is

$$\big((H^{1,0})^* \odot_{J^1(E,n)} (H^{1,0})^*\big) \otimes_{J^1(E,n)} VJ^1(E,n).$$

For $r > 1$ we have the following commutative diagram

$$\begin{tikzcd}
J^{r+1}(E,n) \arrow[r, "D^{r+1}"] \arrow[d, "\pi_{r+1,r}"'] & (H^{1,0})^* \otimes_{J^r(E,n)} TJ^r(E,n) \arrow[d, "\mathrm{id} \otimes T\pi_{r,r-1}"] \\
J^r(E,n) \arrow[r, "D^r"] & (H^{1,0})^* \otimes_{J^r(E,n)} TJ^{r-1}(E,n)
\end{tikzcd}$$

and we get that $\pi_{r+1,r}$ is an affine bundle by a similar reasoning. It is now easy to obtain the associated vector bundle by induction. $\square$

## 1.5 Forms on jets

Here we study the spaces of forms on jets in view of the formulation of the finite order $\mathcal{C}$-spectral sequence. We introduce spaces of *contact* and *horizontal* forms. Contact forms vanish when calculated on any prolonged submanifold; horizontal forms are forms which do not contain contact factors. We show that horizontal forms have a special polynomial structure which has already studied from an intrinsic viewpoint in the case of jets of fibrings [38, 47]. We prove that such a structure is present also on horizontal forms on zero-order jets.

Note that such a polynomial structure has been introduced and studied in a coordinate (or local) fashion so far (*hyperjacobians*, [36], and [2], [3, chap. 4]); in this paper we find it as a by-product of our geometric structure. This also provides a better understanding of its transformation laws.

We denote by $\mathcal{F}_r$ the algebra $C^\infty(J^r(E,n))$. For $k \geq 0$ we denote by $\Lambda^k_r$ the $\mathcal{F}_r$-module of $k$-forms on $J^r(E,n)$. We also set $\Lambda^*_r = \bigoplus_k \Lambda^k_r$. We introduce the submodule of $\Lambda^k_r$ of the *contact forms*

$$\mathcal{C}^1\Lambda^k_r \stackrel{\text{def}}{=} \{\, \alpha \in \Lambda^k_r \mid (j_r L)^*\alpha = 0 \quad \text{for each submanifold } L \subset E \,\}.$$

Contact forms are clearly the annihilators of the Cartan distribution (see remark 3). We set $\mathcal{C}^1\Lambda^*_r = \bigoplus_k \mathcal{C}^1\Lambda^k_r$. Moreover, we define $\mathcal{C}^p\Lambda^*_r$ as the $p$-th exterior power of $\mathcal{C}^1\Lambda^*_r$. Of course, $\mathcal{C}^p\Lambda^k_r = \mathcal{C}^p\Lambda^*_r \cap \Lambda^k_r$.

Next we introduce the $\mathcal{F}_{r+1}$-module $\Lambda^k_{r+1,r}$ of sections of the bundle $\bigwedge^k(T^{r+1,r})^*$. This module is formed by $k$-forms along $\pi_{r+1,r}$, *i.e.*, $k$-forms on $J^r(E,n)$ with coefficients in $\mathcal{F}_{r+1}$.

We also consider the $\mathcal{F}_{r+1}$-module $\mathcal{H}^k_{r+1,r}$ of *pseudo-horizontal* $k$-forms, *i.e.*, sections of the bundle $\bigwedge^k(H^{r+1,r})^*$.

Now we introduce an operation that allows us to extract from any form $\alpha \in \Lambda^k_r$ its 'horizontal part'.

**Definition 3.** Let $q \in \mathbb{N}$. *Horizontalisation* is the map

$$h^{0,q}\colon \Lambda^q_r \longrightarrow \mathcal{H}^q_{r+1,r}, \quad \alpha \longmapsto (\wedge^q(D^{r+1})^*) \circ (\pi^*_{r+1,r}\alpha).$$

Of course, the above operation is trivial if $q > n$. Horizontalisation is well-defined because $\pi^*_{r+1,r}(\alpha)$ has values in $\bigwedge^q(T^{r+1,r})^*$, which is a subbundle of $\bigwedge^q(T^*J^{r+1}(E,n))$ (see subsection 1.3). If $\alpha \in \Lambda^1_r$ has the coordinate expression $\alpha = \alpha_\lambda dx^\lambda + \alpha^{\boldsymbol{\sigma}}_i du^i_{\boldsymbol{\sigma}}$ $(0 \le |\boldsymbol{\sigma}| \le r)$, then

$$h^{0,1}(\alpha) = (\alpha_\lambda + u^i_{\boldsymbol{\sigma},\lambda}\alpha^{\boldsymbol{\sigma}}_i)\,\overline{dx}^\lambda.$$

In general if $\alpha \in \Lambda^q_r$, then we have the coordinate expression

$$\alpha = \alpha^{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_h}_{i_1\ldots i_h}{}_{\lambda_{h+1}\ldots\lambda_q} du^{i_1}_{\boldsymbol{\sigma}_1} \wedge \cdots \wedge du^{i_h}_{\boldsymbol{\sigma}_h} \wedge dx^{\lambda_{h+1}} \wedge \cdots \wedge dx^{\lambda_q}, \tag{10}$$

where $0 \le h \le q$. Hence

$$h^{0,q}(\alpha) = u^{i_1}_{\boldsymbol{\sigma}_1,\lambda_1} \ldots u^{i_h}_{\boldsymbol{\sigma}_h,\lambda_h} \alpha^{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_h}_{i_1\ldots i_h}{}_{\lambda_{h+1}\ldots\lambda_q} \overline{dx}^{\lambda_1} \wedge \cdots \wedge \overline{dx}^{\lambda_q}. \tag{11}$$

Let us introduce the $\mathcal{F}_r$-module $\overline{\Lambda}^q_r \overset{\text{def}}{=} \operatorname{im} h^{0,q}$. It is easy to realize from the above coordinate expressions that, if $r \ge 1$, then $\overline{\Lambda}^q_r$ is made by elements of $\mathcal{H}^q_{r+1,r}$ whose coefficients are fibred polynomials of degree $q$ in the highest order variables (i. e., $u^i_{\boldsymbol{\sigma}}$ with $|\boldsymbol{\sigma}| = r+1$). Of course, this feature is intrinsic due to the affine structure of $\pi_{r+1,r}$. But $\overline{\Lambda}^q_r$ does not coincide with the space of *all* such polynomial forms: indeed, not all polynomial forms come from the horizontalisation of a form on a jet space, unless $n = 1$. From (11) it follows that the coefficients of monomials present a skew-symmetry with respect to the exchange of pairs ${}^{i}_{\boldsymbol{\sigma}}$ and ${}^{j}_{\boldsymbol{\tau}}$. This property appears analogously in the case of jets of fibrings, see [47, 48, 49].

The case $r = 0$ needs a special attention. In a similar way to (3), we realize that $\overline{dy}^\mu = (J^\mu_\lambda + J^\mu_i u^i_\lambda)\overline{dx}^\lambda$. Combining this formula with (11) we deduce that the set of sections of the bundle $\overline{\Lambda}^q_0$ admit a subspace of sections with polynomial coefficients. So, even if $\pi_{1,0}$ *is not* an affine bundle, $\overline{\Lambda}^q_0$ is a subspace of the space of forms with polynomial coefficients of degree $q$ in $u^i_\lambda$.

*Remark* 4. The above polynomial structure has been studied in a coordinate setting in [36]. The alternated sum of monomials of the type $u^{i_1}_{\boldsymbol{\sigma}_1,\lambda_1} \ldots u^{i_h}_{\boldsymbol{\sigma}_h,\lambda_h}$ is called *hyper-jacobian*. We stress, however, that such a structure emerges naturally by virtue of the geometric properties of our scheme, hence it is a *global* property, in contrast with the analysis of [2] and [3, chap. 4], where such a polynomial structure is treated as a local property.

Then, we study contact forms and their relationship with horizontalisation.

**Lemma 2.** *Let $\alpha \in \Lambda^q_r$, with $0 \leq q \leq n$. We have $(j_r L)^*(\alpha) = (j_{r+1}L)^*(h^{0,q}(\alpha))$.*

*Proof.* This follows from the expression of $h^{0,q}$ (definition 3) and the fact that $D^{r+1}$ is the identity on the image of $Tj_rL$, *i.e.*, on $TL^{(r)}$. $\square$

As an obvious consequence of the previous lemma, we have

$$\mathcal{C}^1\Lambda^q_r = \ker h^{0,q} \quad \text{if} \quad 0 \leq q \leq n, \qquad \mathcal{C}^1\Lambda^q_r = \Lambda^q_r \quad \text{if} \quad q > n. \tag{12}$$

Moreover, the forms in $\mathcal{C}^1\Lambda^q_r$ can be characterised as follows:

$$\alpha \in \mathcal{C}^1\Lambda^q_r \quad \Leftrightarrow \quad \pi^*_{r+1,r}(\alpha) \in \operatorname{im}((\omega^{r+1})^* \wedge \mathrm{id}). \tag{13}$$

It turns out that, if $\alpha \in \mathcal{C}^p\Lambda^{p+q}_r$, then we have the coordinate expression

$$\pi^*_{r+1,r}(\alpha) = \omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge \cdots \wedge \omega^{i_p}_{\boldsymbol{\sigma}_p} \wedge \alpha^{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_p}_{i_1\ldots i_p}, \qquad \alpha^{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_p}_{i_1\ldots i_p} \in \pi^*_{r+1,r}\left(\Lambda^q_r\right), \tag{14}$$

where $|\boldsymbol{\sigma}_l| \leq r$ for $l = 1, \ldots, p$. Note that:

1. derivatives of order $r+1$ appear in the above expression in the forms $\omega^{i_l}_{\boldsymbol{\sigma}_l}$ with $|\boldsymbol{\sigma}_l| = r$. It is possible to obtain an expression containing just $r$-th order derivatives by using contact forms of the type $d\omega^{i_l}_{\boldsymbol{\sigma}_l}$ with $|\boldsymbol{\sigma}_l| = r-1$; see [28];

2. in the case $q = 0$ it can be proved (for $p = 1$, see [28]; in the general case the argument is similar, see [49]) that $|\boldsymbol{\sigma}_l| \leq r-1$ for $l = 1, \ldots, p$ in the above coordinate expression.

From the above consideration it follows that the horizontalisation allows us to discard contact components from a form. And contact components produce no contribution to action-like functionals (section 4). Moreover, we shall see that the first term of the $\mathcal{C}$-spectral sequence is made by quotients of $p$-contact forms by $(p+1)$-contact forms (section 2), so that it is important to be able to discard $(p+1)$-contact factors from a $p$-contact form. Hence, for future purposes, we introduce the following *partial horizontalisation* map

$$h^{p,q} \colon \Lambda^{p+q}_r \longrightarrow \Lambda^p_{r+1,r} \otimes \overline{\Lambda}^q_r, \quad \alpha \longmapsto (\wedge^p \mathrm{id} \otimes \wedge^q D^{r+1*}) \circ (\pi^*_{r+1,r}\alpha). \tag{15}$$

The action of $h^{p,q}$ on decomposable forms is

$$h^{p,q}(\alpha_1 \wedge \ldots \wedge \alpha_{p+q}) = \frac{1}{p!\,q!} \sum_{\sigma \in S_{p+q}} |\sigma|\, \pi^*_{r+1,r}(\alpha_{\sigma(1)} \wedge \ldots \wedge \alpha_{\sigma(p)}) \otimes h^{0,q}(\alpha_{\sigma(p+1)} \wedge \ldots \wedge \alpha_{\sigma(p+q)}),$$

where $S_{p+q}$ is the set of permutations of $p+q$ elements.

# 2 Spectral sequence

In this section we present a new finite order approach to $\mathcal{C}$-spectral sequence on the jets of submanifolds of order $r$. This approach has been directly inspired by Vinogradov's $\mathcal{C}$-spectral sequence on infinite order jets [43, 44, 45]. A first analysis of the finite order $\mathcal{C}$-spectral sequence already appeared in [33]. We complete that exposition with the new coordinate expressions (16), (2) and the relations between our differentials and the standard horizontal differential $d_H$. We also point out in remark 7 that our methods yield a generalisation of results previously obtained in [28] and [16].

The module $\Lambda^k_r$ is filtered by the submodules $\mathcal{C}^p\Lambda^k_r$; namely, we have the obvious *finite* filtration

$$\Lambda^k_r \stackrel{\text{def}}{=} \mathcal{C}^0\Lambda^k_r \supset \mathcal{C}^1\Lambda^k_r \supset \cdots \supset \mathcal{C}^p\Lambda^k_r \supset \cdots \supset \mathcal{C}^I\Lambda^k_r \supset \mathcal{C}^{I+1}\Lambda^k_r = \{0\},$$

where $I$ is the codimension of the Cartan distribution (see [7]). The filtration is stable with respect to the differential of forms, *i.e.*, $d(\mathcal{C}^p\Lambda^k_r) \subset \mathcal{C}^p\Lambda^{k+1}_r$. We say that the above graded filtration is the *$\mathcal{C}$-filtration* on the jet space of order $r$.

The $\mathcal{C}$-filtration gives rise to a spectral sequence $(E^{p,q}_N, e_N)_{N,p,q\in\mathbb{N}}$ in the usual way. We recall that a spectral sequence is a sequence of differential Abelian groups where each term is the cohomology of the previous one, with the exception of $E^{*,*}_0$, which, in our case, is the set of quotients between consecutive terms of the $\mathcal{C}$-filtration (see, *e.g.*, [26, 48]).

**Definition 4.** We call the above spectral sequence the *Vinogradov's $\mathcal{C}$-spectral sequence of (finite) order* $r$ on $E$.

Next goal is to recall the description of all terms in the finite order $\mathcal{C}$-spectral sequence made in [33].

We recall that $E^{p,q}_0 \equiv \mathcal{C}^p\Lambda^{p+q}_r / \mathcal{C}^{p+1}\Lambda^{p+q}_r$. Generalising the equalities (12) to partial horizontalisation we obtain the following result.

**Lemma 3.**

$$\mathcal{C}^{p+1}\Lambda^{p+q}_r = \ker h^{p,q} \quad \textit{if} \quad q \leq n, \qquad \mathcal{C}^{p+1}\Lambda^{p+q}_r = \Lambda^{p+q}_r \quad \textit{if} \quad q > n.$$

As a consequence, we are able to express any equivalence class of $E_0$ with a distinguished form. To proceed with our investigation, we need to describe the target space of partial horizontalisation of contact forms. Taking into account (14), we introduce the space $\mathcal{C}^p\Lambda^p_{r,r+1} \subset \mathcal{C}^p\Lambda^p_{r+1}$ of contact forms with coefficients in $\mathcal{F}_r$. This space can be characterised in an intrinsic way as follows. A form $\gamma$ is in $\mathcal{C}^p\Lambda^p_{r,r+1}$ if and only if $\gamma = \pi^*_{r+1,r}(i_{X_1} \cdots i_{X_q}\gamma')$, where $\gamma' \in \mathcal{C}^p\Lambda^{p+q}_r$ and $X_1, \ldots, X_q \colon J^1(E,n) \longrightarrow H^{1,0}$. Of course, the action of $\gamma'$ on the vector fields is obtained through the isomorphism (7).

**Proposition 1 (Computation of $E_0$, [33]).** *Let $q > 0$. Then, the restriction of $h^{p,q}$ to $\mathcal{C}^p\Lambda^{p+q}_r$ yields the isomorphism*

$$E^{p,q}_0 = \mathcal{C}^p\Lambda^{p+q}_r / \mathcal{C}^{p+1}\Lambda^{p+q}_r \longrightarrow \mathcal{C}^p\Lambda^p_{r,r+1} \otimes \bar{\Lambda}^q_r, \qquad [\alpha] \longmapsto h^{p,q}(\alpha).$$

Obviously, if $q = 0$ then $E_0^{0,0} = \mathcal{F}_r$, and $E_0^{p,0} = \mathcal{C}^p\Lambda^p_r$ for $p > 0$. We set $\bar{d} \overset{\text{def}}{=} e_0$; we have $\bar{d}(h^{p,q}(\alpha)) = h^{p,q+1}(d\alpha)$. Hence, the bigraded complex $(E_0, e_0)$ is isomorphic to the sequence of complexes $(\mathcal{C}^p\Lambda^p_{r,r+1} \otimes \bar{\Lambda}^*_r, \bar{d})_{p\in\mathbb{N}}$. The complexes have finite length $n$, and the sequence is trivial for $p > I$. For $p = 0$ this is just the *horizontal de Rham complex of order* $r$ (see [42] for the infinite order version).

We observe that the coordinate expression of $\bar{\alpha} \in \mathcal{C}^p\Lambda^p_{r,r+1} \otimes \bar{\Lambda}^q_r$ is

$$
\begin{aligned}
\bar{\alpha} &= \bar{\alpha}^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_p}_{i_1\cdots i_p\ \lambda_1\cdots\lambda_q}\omega^{i_1}_{\boldsymbol{\sigma}_1} \wedge\cdots\wedge \omega^{i_p}_{\boldsymbol{\sigma}_p} \otimes \overline{dx}^{\lambda_1} \wedge\cdots\wedge \overline{dx}^{\lambda_q}, \\
\bar{\alpha}^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_p}_{i_1\cdots i_p\ \lambda_1\cdots\lambda_q} &= \alpha^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_p\ \boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_l}_{i_1\cdots i_p\ j_1\cdots j_l\ \lambda_{l+1}\cdots\lambda_q} u^{j_1}_{\boldsymbol{\tau}_1,\lambda_1}\cdots u^{j_l}_{\boldsymbol{\tau}_l,\lambda_l},
\end{aligned}
\tag{16}
$$

with $0 \leq |\boldsymbol{\sigma}_k| \leq r$, $|\boldsymbol{\tau}_h| = r$, $0 \leq l \leq q$ and $\alpha^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_p\ \boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_l}_{i_1\cdots i_p\ j_1\cdots j_l\ \lambda_{l+1}\cdots\lambda_q} \in \mathcal{F}_r$.

**Proposition 2.** *Let $\bar{\alpha} \in \bar{\Lambda}^q_r$. Then, we have the coordinate expression*

$$
\bar{d}\bar{\alpha} = D_\lambda(\alpha^{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_l}_{j_1\cdots j_l\ \lambda_{l+1}\cdots\lambda_q}) u^{j_1}_{\boldsymbol{\tau}_1,\lambda_1}\cdots u^{j_l}_{\boldsymbol{\tau}_l,\lambda_l}\overline{dx}^{\lambda} \wedge \overline{dx}^{\lambda_1} \wedge\cdots\wedge \overline{dx}^{\lambda_q}.
\tag{17}
$$

*Proof.* A form $\alpha \in \Lambda^q_r$ such that $h^{0,q}(\alpha) = \bar{\alpha}$ has the coordinate expression

$$
\alpha = \alpha^{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_l}_{j_1\cdots j_l\ \lambda_{l+1}\cdots\lambda_q} du^{j_1}_{\boldsymbol{\tau}_1} \wedge\cdots\wedge du^{j_l}_{\boldsymbol{\tau}_l} \wedge dx^{\lambda_{l+1}} \wedge\cdots\wedge dx^{\lambda_q}.
$$

From the expression of $d\alpha$ we easily get the result. □

In the case that $\alpha$ belongs to $\mathcal{C}^p\Lambda^p_{r,r+1} \otimes \bar{\Lambda}^q_r$ for $p \geq 1$, one has also to differentiate contact forms (see (16)). We have $d\omega^i_{\boldsymbol{\sigma}} = -du^i_{\boldsymbol{\sigma},\lambda} \wedge dx^\lambda$, hence the differentiation when $|\boldsymbol{\sigma}| = r$ yields a form on an higher order jet. So, to produce a coordinate expression for $\bar{d}\bar{\alpha}$ we need a form $\alpha$ such that $h^{p,q}(\alpha) = \bar{\alpha}$ and whose expression use $d\omega^i_{\boldsymbol{\rho}}$ with $|\boldsymbol{\rho}| = r-1$, instead of $\omega^i_{\boldsymbol{\sigma}}$ with $|\boldsymbol{\sigma}| = r$. Such an expression can be found in [28] (see also (14)).

*Remark* 5. It is important to stress that the differential $\bar{d}$ of the previous complexes does not coincide with the *horizontal differential* $d_H$ (denoted by $\widehat{d}$ in [7]; see also [39]) used in the infinite order formalism. The action of $d_H$ is characterised by the coordinate expressions $d_H f = D_\lambda f\, dx^\lambda$, for $f \in \mathcal{F}$, $d_H\, dx^\lambda = 0$, $d_H\omega^i_{\boldsymbol{\sigma}} = -\omega^i_{\boldsymbol{\sigma},\lambda} \wedge dx^\lambda$. Hence, $\bar{d}$ does not change the order of jet space while $d_H$ raises the jet order by one.

Moreover, from the above proposition it turns out that $\bar{d}\bar{\alpha} = d_H\bar{\alpha}$ if $\bar{\alpha}$ does not contain derivatives of order $r+1$, *i.e.*, if $\bar{\alpha} \in \mathcal{C}^p\Lambda^p_{r-1,r} \otimes \bar{\Lambda}^q_r \subset \mathcal{C}^p\Lambda^p_{r,r+1} \otimes \bar{\Lambda}^q_r$. In other words, $\bar{d}$ and $d_H$ just differ on highest order derivatives. Hence, it is clear that these two operators coincide in the direct limit, *i.e.*, on infinite order jets.

We recall that $E_1 = H(E_0)$, where the cohomology is taken with respect to $\bar{d}$. We start by determining the term $E_1^{p,n}$.

**Proposition 3 ([33]).** *We have the diagram*

$$
\begin{tikzcd}
 & 0 & 0 & & 0 & \\
0 \arrow[r] & \bar{\Lambda}^n_r/\bar{d}(\bar{\Lambda}^{n-1}_r) \arrow[u] \arrow[r, "e_1"] & E_0^{1,n}/\bar{d}(E_0^{1,n-1}) \arrow[u] \arrow[r, "e_1"] & \dots \arrow[r, "e_1"] & E_0^{p,n}/\bar{d}(E_0^{p,n-1}) \arrow[u] \arrow[r, "e_1"] & \dots \\
 & \bar{\Lambda}^n_r \arrow[u] \arrow[ur, "\widetilde{e_1}"'] & E_0^{1,n} \arrow[u] & \dots & E_0^{p,n} \arrow[u] & \dots \\
 & \dots \arrow[u, "\bar{d}"'] & \dots \arrow[u, "\bar{d}"'] & \dots & \dots \arrow[u, "\bar{d}"'] & \dots
\end{tikzcd}
\tag{18}
$$

*where $E_1^{*,n}$ is the top row, and $e_1([h^{p,n}(\alpha)]) = [h^{p+1,n}(d\alpha)]$.*

For a proof, see [33]. In the above diagram, the maps without a label are either trivial of quotient projections. The map $\widetilde{e_1}$ is just the composition of $e_1$ with the quotient projection $\bar{\Lambda}^n_r \longrightarrow \bar{\Lambda}^n_r/\bar{d}(\bar{\Lambda}^{n-1}_r)$.

*Remark* 6. In order to match a similar convention in the infinite order formulation, in (18) we can replace $\bar{d}$ with $(-1)^p\bar{d}$.

In order to determine $E_1^{p,q}$ with $q < n$ we need some preliminary results. First of all, the sequence

$$
0 \longrightarrow \mathcal{C}^p\Lambda^p_r \xrightarrow{d} \mathcal{C}^p\Lambda^{p+1}_r \xrightarrow{d} \dots \xrightarrow{d} \mathcal{C}^p\Lambda^{p+n-1}_r \xrightarrow{d} \dots \tag{19}
$$

is exact up to the term $\mathcal{C}^p\Lambda^{p+n-1}_r$ [33]. The $\mathcal{C}$-spectral sequence converges to the de Rham cohomology of $J^r(E,n)$, because it is a first quadrant spectral sequence [33]. Moreover, the cohomology of $J^r(E,n)$ is equal to the cohomology of $J^1(E,n)$ due to the topological triviality of the fibres. Summing up these facts, we have

1. $E_1^{0,q} = H^q(J^1(E,n))$, for $q \neq n$;
2. $E_2^{p,n} = H^p(J^1(E,n))$, for $p \geq 1$;
3. $E_1^{p,q} = 0$ for $q \neq n$ and $p \neq 0$.

This completes the description of all terms of the finite order $\mathcal{C}$-spectral sequence. The interested reader can find details and proofs in [33].

*Remark* 7. A partial result in the direction of the computation of the finite order $\mathcal{C}$-spectral sequence on jets of submanifolds already appeared in [16]. In that paper the term $E_1^{0,q}$ was computed with interesting intrinsic techniques (Koszul complexes), by analogy with the infinite order case. This term is known as *horizontal cohomology* [42].

However, we compute *all* terms of the finite order $\mathcal{C}$-spectral sequence. Our method is a direct generalisation of the argument that Krupka used to compute $E_1^{0,q}$ in his setting on jets of fibrings [28]. This is a direct consequence of the exactness of the sequence (19) for $p = 1$, which is proved in [28] through a 'vertical' Poincaré lemma and elementary facts from sheaf theory, from which the results in [16] follow.

# 3 Green-Vinogradov formula and the finite order variational sequence

The diagram (18) contains a further complex which is of fundamental importance for the calculus of variations: the variational sequence.

**Definition 5.** The complex

$$\dots \xrightarrow{\bar{d}} \bar{\Lambda}_r^{n-1} \xrightarrow{\bar{d}} \bar{\Lambda}_r^{n} \xrightarrow{\tilde{e}_1} E_0^{1,n}/\bar{d}(E_0^{1,n-1}) \xrightarrow{e_1} E_0^{2,n}/\bar{d}(E_0^{2,n-1}) \xrightarrow{e_1} \dots,$$

is said to be the *variational sequence of order $r$*.

Elements of $E_0^{p,n}/\bar{d}(E_0^{p,n-1})$ are said to be *variational $p$-forms of order $r$*.

From the calculations of the terms of the $\mathcal{C}$-spectral sequence it follows that the cohomology of the above complex is isomorphic to the de Rham cohomology of $J^1(E, n)$.

In the last section we shall analyse the relationship between the variational sequence and the calculus of variations; for the moment, we concentrate on another problem. Namely, variational forms are equivalence classes; we are going to prove that each equivalence class can be represented by a distinguished form. To this purpose, the most important tool is the Green–Vinogradov formula, which is the geometric analogue of the integration by parts.

## 3.1 Green–Vinogradov formula

In this subsection we first exhibit a natural isomorphism between the module of contact forms and a suitable space of differential operators. This allows us to 'import' the theory of adjoint operators and the Green–Vinogradov formula in our setting (see [7, 26] for introductory expositions).

Here, we also show that this procedure does not have a finite order analogue. So, we are forced to use the infinite order theory to provide distinguished representatives for equivalence classes in the variational sequence. This is well-known in the infinite order case, but its application to the finite-order case is new in the case of jets of submanifolds, up to partial results contained in [33].

We observe that our representation solves a problem which was left open in [16] (remark 8).

Let $P$, $Q$ be projective modules over an $\mathbb{R}$-algebra $A$. We recall (see [4] for instance) that a *linear differential operator* of order $k$ is an $\mathbb{R}$-linear map $\Delta \colon P \longrightarrow Q$ such that

$$[\delta_{a_0}, [\dots, [\delta_{a_k}, \Delta] \dots]] = 0$$

for all $a_0, \dots, a_k \in A$. Here, square brackets stand for commutators and $\delta_{a_i}$ is the multiplication morphism by $a_i$. Of course, linear differential operators of order zero are morphisms of modules. This definition can be generalised to differential operators in several arguments.

We are interested in differential operators between $\mathcal{F}_r$-modules whose expressions contain total derivatives instead of partial ones. Unfortunately, such operators raise the jet order by one, so we are lead to extend the above definition in an obvious way to the case in which $P$ is an $\mathcal{F}_r$-module and $Q$ is an $\mathcal{F}_s$-module, with $r \leq s$ (taking into account the inclusion $\mathcal{F}_r \subset \mathcal{F}_s$).

Now, let $k \leq s - r$; we say that a differential operator $\Delta \colon P \longrightarrow Q$ of order $k$ is *$\mathcal{C}$-differential* if it can be restricted to manifolds of the form $L^{(r)}$ and $L^{(s)}$. In local coordinates, $\mathcal{C}$-differential operators have the matrix form $(a^{\boldsymbol{\sigma}}_{ij} D_{\boldsymbol{\sigma}})$, where $a^{\boldsymbol{\sigma}}_{ij} \in \mathcal{F}_s$, $D_{\boldsymbol{\sigma}} = D_{\sigma_1} \circ \cdots \circ D_{\sigma_h}$, and $|\boldsymbol{\sigma}| = h \leq k$. The space of such operators is denoted by $\mathcal{C}\mathrm{Diff}_k(P, Q)$. We shall deal with spaces of antisymmetric $\mathcal{C}$-differential operators of $l$ arguments in $P$, which we denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(l)\,k}(P, Q)$.

Next, we consider sections of pseudo-vertical bundles. Let $X$ be a vector field on $J^r(E, n)$. Its *vertical part* $X_V$ is the section of $V^{r+1,r}$ defined by $X_V \overset{\mathrm{def}}{=} \omega^{r+1}(X \circ \pi_{r+1,r})$. If $X = X^\lambda \partial/\partial x^\lambda + X^i_{\boldsymbol{\sigma}} \partial/\partial u^i_{\boldsymbol{\sigma}}$, then $X_V = (X^i_{\boldsymbol{\sigma}} - u^i_{\boldsymbol{\sigma},\lambda} X^\lambda)[\partial/\partial u^i_{\boldsymbol{\sigma}}]$.

For $r \geq 1$ let us denote the $\mathcal{F}_r$-module of bundle morphisms $\varphi \colon J^r(E, n) \longrightarrow V^{1,0}$ over $\mathrm{id}_{J^1(E,n)}$ by $\varkappa_r$. Of course $\varkappa_1$ is just the set of the sections of $V^{1,0}$. We also define $\varkappa_0 \subset \varkappa_1$ as the subset of vertical parts of vector fields on $E$. Any $\varphi \in \varkappa_r$ can be uniquely prolonged to an *evolutionary vector field* $\ni_\varphi \colon J^{r+s}(E, n) \longrightarrow V^{s+1,s}$ (see [7]). In coordinates, if $\varphi = \varphi^i\,[\partial/\partial u^i]$, then $\ni_\varphi = D_{\boldsymbol{\sigma}} \varphi^i\,[\partial/\partial u^i_{\boldsymbol{\sigma}}]$, with $|\boldsymbol{\sigma}| \leq s$.

**Proposition 4 ([33, 49]).** *We have the natural isomorphism*

$$\mathcal{C}^p\Lambda^p_{r,r+1} \otimes \bar{\Lambda}^q_r \longrightarrow \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^q_r),\ \bar{\alpha} \longmapsto \nabla_{\bar{\alpha}}(\varphi_1, \dots, \varphi_p) = \frac{1}{p!}\, i_{\ni_{\varphi_1}}(\cdots i_{\ni_{\varphi_p}}(\bar{\alpha}) \cdots).$$

The above proposition can be proved by analogy with the infinite order case (see [7]). The numerical factor is put in order to have simpler coordinate expressions: from (16) we have

$$\nabla_{\bar{\alpha}}(\varphi_1, \dots, \varphi_p) = \alpha^{\boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_p}_{i_1 \cdots i_p\ \lambda_1 \cdots \lambda_q} D_{\boldsymbol{\sigma}_1} \varphi^{i_1}_1 \cdots D_{\boldsymbol{\sigma}_p} \varphi^{i_p}_p\, \overline{dx}^{\lambda_1} \wedge \ldots \wedge \overline{dx}^{\lambda_q}. \tag{20}$$

Let $\mathcal{F}$ be the direct limit of the chain $\cdots \subset \mathcal{F}_r \subset \mathcal{F}_{r+1} \subset \cdots$ of injections given by pull-back. In other words, $\mathcal{F}$ is the space of all functions on any finite order jet. In what follows, we shall drop the order index $r$ to indicate spaces obtained in similar ways. We say that $P$ is a *horizontal module* if $P$ is an $\mathcal{F}$-module obtained as direct limit of an ascending chain of injections of $\mathcal{F}_r$-modules. As an example, the injections $\varkappa_i \hookrightarrow \varkappa_{i+1}$ yield a sequence whose direct limit, denoted by $\varkappa$, is a horizontal module. See [7] for connections between $\varkappa$ and symmetries of differential equations.

Let $p > 0$; for any horizontal module $P$ we have the complexes $(\mathcal{C}\mathrm{Diff}_{(p)}(P, \bar{\Lambda}^*), w)$, where $w(\nabla) = d_H \circ \nabla$. In [7] it is proved that the only non-vanishing cohomology group of such complexes is the $n$-th, which is equal to $\mathcal{C}\mathrm{Diff}_{(p-1)}(P, \widehat{P})$, where

$\widehat{P} \overset{\text{def}}{=} \operatorname{Hom}_{\mathcal{F}}(P, \bar{\Lambda}^n)$. Any $\mathcal{C}$-differential operator $\Delta\colon P \longrightarrow Q$ induces a map of the corresponding $w$-complexes, hence a cohomology map $\Delta^*\colon \widehat{Q} \longrightarrow \widehat{P}$. It fulfils the *Green-Vinogradov* formula [43, 44]

$$\widehat{q}(\Delta(p)) - (\Delta^*(\widehat{q}))(p) = d_H \omega_{p,\widehat{q}}(\Delta). \tag{21}$$

In coordinates, if $\Delta = \Delta_{ij}^{\boldsymbol{\sigma}} D_{\boldsymbol{\sigma}}$, then $\Delta^* = (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}} \circ \Delta_{ji}^{\boldsymbol{\sigma}}$. One can easily realize the meaning of the above formula by observing that the left-hand side contains no zero order term with respect to $p$, hence it must be the total derivative of a certain $\omega_{p,\widehat{q}}$. The problem of representing variational forms in the infinite order formalism is solved by taking the skew-symmetric part of the cohomology of the complexes $(\mathcal{C}\mathrm{Diff}_{(p)}(P, \bar{\Lambda}^*), w)$ with respect to permutations of the arguments ([43, 44, 45]; see also [7, p. 192]). Namely, let $K_p(\varkappa) \subset \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}(\varkappa, \widehat{\varkappa})$ be the subspace of operators $\nabla$ fulfilling $\nabla(\varphi_1, \dots, \varphi_{p-2})^* = -\nabla(\varphi_1, \dots, \varphi_{p-2})$ for all $\varphi_i \in \varkappa$, $i = 1, \dots, p$. Then, we have the isomorphism

$$I_p\colon \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \bar{\Lambda}^n) / \bar{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \bar{\Lambda}^{n-1})) \longrightarrow K_p(\varkappa), \quad [\Delta] \longmapsto \overline{\Delta}. \tag{22}$$

where $\overline{\Delta}(\varphi_1, \dots, \varphi_{p-1})(\varphi_p) \overset{\text{def}}{=} \Delta(\varphi_1, \dots, \varphi_{p-1})^*(1)(\varphi_p)$ for all $\varphi_i \in \varkappa$, $i = 1, \dots, p$.

Now we devote ourselves to the problem of the representation of the finite order variational sequence. We would like to find representatives of the same order of all objects in the given class. But the following attempts, carried out by analogy with the infinite order case, failed.

1. We would like to reproduce the above scheme in the finite order case. We could generalise finite order $\bar{d}$-complexes of antisymmetric operators to finite order complexes with arguments in a module $P$ over $\mathcal{F}_r$ of operators of any kind. Unfortunately, any $\mathcal{C}$-differential operator $\Delta\colon P \longrightarrow Q$ of order $k > 0$ is a *graded* map of such finite order complexes, in the sense that it raises the order by $k$. Hence, there is no hope to find a version of 'adjoint operator' which is order-preserving.

2. Let $p = 1$ for simplicity. Given $\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^n_r)$ we can look for operators $\bar{\omega}_\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(1)\,r}(\varkappa_0, \bar{\Lambda}^{n-1}_r)$ and $\bar{\Delta} \in \operatorname{Hom}_{\mathcal{F}_r}(\varkappa_0, \bar{\Lambda}^n_r)$ such that $\Delta + \bar{d}(\bar{\omega}_\Delta) = \bar{\Delta}$. Unfortunately, the previous equation always admits a unique solution if we let $\bar{\Delta}$ and $\bar{\omega}_\Delta$ to be defined on an higher order jet. A proof of this statement can be easily achieved by the same argument used by Kolář [23] and the coordinate expression of $\bar{d}$ (proposition 2).

The problem of representation is thus solved by embedding the finite order variational sequence into the infinite order variational sequence and using the standard results of the infinite order theory restricted to the image of the embedding. To this aim, we observe that pull-back allows us to take $\bar{d} = d_H$ (remark 5). Moreover, pull-back is a map of complexes between $\mathcal{C}$-variational sequences of different order [48], hence the direct limit of the $r$-th order $\mathcal{C}$-variational sequence is just the standard infinite order $\mathcal{C}$-variational sequence.

**Theorem 2.** *The representation of each space of finite order variational sequence (in the case $p > 1$) is the image space $K_{p,r}(\varkappa_0) \overset{def}{=} \operatorname{im}(I_p \circ \chi_{p,r})$, where*

$$\chi_{p,r}\colon \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)/\bar{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^{n-1}_r)) \hookrightarrow \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \bar{\Lambda}^n)/\bar{d}(\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\varkappa, \bar{\Lambda}^{n-1}))$$

*is the inclusion induced by pull-back $\pi^*_{r+1,r}$ into the direct limit.*

*Remark* 8. In the paper [16] there is a *conjecture/question* by Griffiths about the existence of natural representatives for $E^{1,n}_1$ in the setting of finite order jets of submanifolds. The above theorem answers constructively to this question. We also provide the coordinate expression of such representatives in the next subsection.

*Remark* 9. We could consider the 'complementary' problem to the representative's one. More precisely, given $\alpha \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)$ we can look for an operator $\omega$ fulfilling $\alpha = \nabla_\alpha + \bar{d}\omega$. Such a section always exists due to the vanishing of the cohomology of $\bar{d}$ on the space where $\omega$ lives.

This problem, for $p = 1$, amounts to search a Poincaré–Cartan form (see [23] for a discussion and its solution). In the case of jets of fibrings, a form of that kind can be determined through a linear symmetric connection on the base manifold. See [1] for the general situation $p > 1$. We expect that the above results hold also in the more general framework of jets of submanifolds. See [17, 18] for a first discussion of the problem.

*Remark* 10. In the case of jets of fibrings the problem of the representation has been faced within Krupka's formulation of variational sequences too. We mention [47], with results up to $p = n + 2$, and [27]. But Krupka's formulation produces the same variational sequence as the finite order $\mathcal{C}$-spectral sequence [48], where the representation problem has been considered in [49] and solved with similar techniques as here.

## 3.2 Coordinate expressions

Here we shall provide coordinate expressions for the representative of variational forms that we found in theorem 2. We stress that such expressions have never been written before for $p \geq 3$ in the case of jets of submanifolds. As a by-product, this provides the well-known interpretation of the variational sequence in terms of the calculus of variations (section 4). We also obtain the coordinate expressions of the differentials of the variational sequence.

An operator $\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{(p)\,r}(\varkappa_0, \bar{\Lambda}^n_r)$ has the expression (see also (16))

$$\Delta(\varphi_1, \dots, \varphi_p) = \Delta^{\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_{p-1} \boldsymbol{\tau}}_{i_1 \dots i_{p-1}\, j} D_{\boldsymbol{\sigma}_1} \varphi^{i_1}_1 \cdots D_{\boldsymbol{\sigma}_{p-1}} \varphi^{i_{p-1}}_{p-1} D_{\boldsymbol{\tau}} \varphi^j_p \,\mathrm{Vol}_n, \tag{23}$$

where $0 \leq |\boldsymbol{\sigma}_a| \leq r$, $0 \leq |\boldsymbol{\tau}| \leq r$, $a = 1, \dots, p-1$, $\Delta^{\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_{p-1} \boldsymbol{\tau}}_{i_1 \dots i_{p-1}\, j}$ is a polynomial with respect to $(r+1)$-st derivatives of distinguished type (it contains hyperjacobians: see (11) and related comments, remark 4 and [38]) with coefficients in $\mathcal{F}_r$ and $\mathrm{Vol}_n \overset{\mathrm{def}}{=} n!\, \overline{dx}^1 \wedge \cdots \wedge \overline{dx}^n$ is a local volume form on any submanifold $L \subset E$ which is

concordant with the given chart. We shall use the following Leibnitz formula for total derivatives (see [39] for the case $k=2$)

$$D_{\boldsymbol{\sigma}}(f_1\cdots f_k)=\sum_{|\boldsymbol{\sigma}_1|+|\boldsymbol{\sigma}_2|+\cdots+|\boldsymbol{\sigma}_k|=|\boldsymbol{\sigma}|}\frac{\boldsymbol{\sigma}!}{\boldsymbol{\sigma}_1!\boldsymbol{\sigma}_2!\cdots\boldsymbol{\sigma}_k!}D_{\boldsymbol{\sigma}_1}f_1\cdots D_{\boldsymbol{\sigma}_k}f_k.$$

We also denote by $(\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)$ the multiindex which is the union of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$.

In the case $p=1$ we have $I_1([\Delta])(\varphi_1)=\overline{\Delta}(\varphi_1)=(-1)^{|\boldsymbol{\tau}|}D_{\boldsymbol{\tau}}\Delta_i^{\boldsymbol{\tau}}\varphi_1^i\ \mathrm{Vol}_n$.

In the case $p=2$ we have

$$\begin{aligned}\overline{\Delta}(\varphi_1)(\varphi_2)&=(-1)^{|\boldsymbol{\tau}|}D_{\boldsymbol{\tau}}\left(\Delta_{i_1\ j}^{\boldsymbol{\sigma}_1\boldsymbol{\tau}}D_{\boldsymbol{\sigma}_1}\varphi_1^{i_1}\right)\varphi_2^j\ \mathrm{Vol}_n\\&=\sum_{0\le|\boldsymbol{\nu}|+|\boldsymbol{\tau}_1|\le r}(-1)^{|(\boldsymbol{\nu},\boldsymbol{\tau}_1)|}\frac{(\boldsymbol{\nu},\boldsymbol{\tau}_1)!}{\boldsymbol{\nu}!\boldsymbol{\tau}_1!}D_{\boldsymbol{\nu}}\Delta_{i_1\ j}^{\boldsymbol{\sigma}_1(\boldsymbol{\nu},\boldsymbol{\tau}_1)}D_{(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)}\varphi_1^{i_1}\varphi_2^j\ \mathrm{Vol}_n\\&=\left(\sum_{\substack{(\boldsymbol{\sigma}_1,\boldsymbol{\tau}_1)=\boldsymbol{\rho}_1\\0\le|\boldsymbol{\rho}_1|\le 2r}}(-1)^{|(\boldsymbol{\nu},\boldsymbol{\tau}_1)|}\frac{(\boldsymbol{\nu},\boldsymbol{\tau}_1)!}{\boldsymbol{\nu}!\boldsymbol{\tau}_1!}D_{\boldsymbol{\nu}}\Delta_{i_1\ j}^{\boldsymbol{\sigma}_1(\boldsymbol{\nu},\boldsymbol{\tau}_1)}\right)D_{\boldsymbol{\rho}_1}\varphi_1^{i_1}\varphi_2^j\ \mathrm{Vol}_n,\end{aligned}\tag{24}$$

where the last passage follows after renaming multiindexes and rearranging terms.

In the case $p\ge 3$ we proceed by analogy with the case $p=2$ with only the change of the length of the product on which to apply Leibnitz rule. We have

(25)

$$\begin{aligned}&\overline{\Delta}(\varphi_1,\dots,\varphi_{p-1})(\varphi_p)=\\&\quad=(-1)^{|\boldsymbol{\tau}|}D_{\boldsymbol{\tau}}(\Delta_{i_1\cdots i_{p-1}\ j}^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_{p-1}\boldsymbol{\tau}}D_{\boldsymbol{\sigma}_1}\varphi_1^i\cdots D_{\boldsymbol{\sigma}_{p-1}}\varphi_{p-1}^{i_{p-1}})\varphi_p^j\ \mathrm{Vol}_n\\&\quad=\sum_{0\le|\boldsymbol{\nu}|+|\boldsymbol{\tau}_1|+\cdots+|\boldsymbol{\tau}_{p-1}|\le r}(-1)^{|(\boldsymbol{\nu},\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_{p-1})|}\frac{(\boldsymbol{\nu},\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_{p-1})!}{\boldsymbol{\nu}!\boldsymbol{\tau}_1!\cdots\boldsymbol{\tau}_{p-1}!}D_{\boldsymbol{\nu}}\Delta_{i_1\cdots i_{p-1}\ j}^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_{p-1}(\boldsymbol{\nu},\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_{p-1})}\times\\&\qquad\quad\times D_{(\boldsymbol{\tau}_1,\boldsymbol{\sigma}_1)}\varphi_1^{i_1}\cdots D_{(\boldsymbol{\tau}_{p-1},\boldsymbol{\sigma}_{p-1})}\varphi_{p-1}^{i_{p-1}}\varphi_p^j\ \mathrm{Vol}_n\\&\quad=\left(\sum_{\substack{(\boldsymbol{\sigma}_i,\boldsymbol{\tau}_i)=\boldsymbol{\rho}_i,\ 0\le|\boldsymbol{\rho}_i|\le 2r,\\1\le i\le p-1}}(-1)^{|(\boldsymbol{\nu},\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_{p-1})|}\frac{(\boldsymbol{\nu},\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_{p-1})!}{\boldsymbol{\nu}!\boldsymbol{\tau}_1!\cdots\boldsymbol{\tau}_{p-1}!}D_{\boldsymbol{\nu}}\Delta_{i_1\cdots i_{p-1}\ j}^{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_{p-1}(\boldsymbol{\nu},\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_{p-1})}\right)\times\\&\qquad\quad\times D_{\boldsymbol{\rho}_1}\varphi_1^{i_1}\cdots D_{\boldsymbol{\rho}_{p-1}}\varphi_{p-1}^{i_{p-1}}\varphi_p^j\ \mathrm{Vol}_n\,.\end{aligned}$$

Now, we shall derive from proposition 3 the expression of the differentials $\tilde{e}_1$ and $e_p$, $p>1$ (the expression of $\bar{d}$ has been derived in proposition 2).

*Remark* 11. In general, given a form $\nabla\in K_{p,r}(\varkappa_0)$ it is rather difficult to find a form $\alpha\in\Lambda_r^{p+q}$ such that $\nabla=I_p([h^{p,n}(\alpha)])$. The commutativity of inclusions between variational

sequences of different orders implies that, locally, $e_p(\nabla_{\bar{\alpha}}) = [d\bar{\alpha}]$ (with the symbols of proposition 4): this last expression is much easier to be computed. Note that $d\bar{\alpha}$ has to be meant as follows: consider (16) as a form on a jet space, removing restriction bars, then take the ordinary differential. Of course, the resulting expression has an intrinsic meaning.

If $\lambda \in \bar{\Lambda}^n_r$ has the coordinate expression $\lambda = \lambda_0 \operatorname{Vol}_n$, then

$$\tilde{e}_1(\lambda)(\varphi_1) = (-1)^{|\boldsymbol{\tau}|} D_{\boldsymbol{\tau}} \left( \frac{\partial}{\partial u^i_{\boldsymbol{\tau}}} \lambda_0 \right) \varphi^i_1 \operatorname{Vol}_n . \tag{26}$$

If $\nabla_{\bar{\alpha}} \in K_{p,r}(\varkappa_0)$, then

$$\begin{aligned} e_p(\nabla_{\bar{\alpha}})(\varphi_1, \dots, \varphi_{p+1}) &= I_p(d\bar{\alpha})(\varphi_1, \dots, \varphi_{p+1}) \\ &= I_p \left( \frac{\partial}{\partial u^{i_1}_{\boldsymbol{\sigma}_1}} \nabla^{\boldsymbol{\sigma}_2 \dots \boldsymbol{\sigma}_{p+1}}_{i_2 \dots i_{p+1}} D_{\boldsymbol{\sigma}_1} \varphi^{i_1}_1 \cdots D_{\boldsymbol{\sigma}_{p+1}} \varphi^{i_{p+1}}_{p+1} \operatorname{Vol}_n \right) . \end{aligned} \tag{27}$$

The coordinate expression of $e_p$ is obtained by a straightforward substitution of the coefficients $\partial/\partial u^{i_1}_{\boldsymbol{\sigma}_1} \Delta^{\boldsymbol{\sigma}_2 \dots \boldsymbol{\sigma}_p}_{i_2 \dots i_p\, j}$ into (25).

# 4 Variational principles and $\mathcal{C}$-spectral sequence

In this section we develop a formalism for the calculus of variations in an intrinsic geometrical setting, in the general case of $n$ independent variables. Such a construction already exists for jets of sections (see [22, 29, 30, 39, 41], for example).

In the case of jets of submanifolds, we found mainly two approaches in literature: the *parametric* approach (see [8, 9, 17, 18] for the general case of $n$ arbitrary; the approach dates back to Carathéodory and earlier for the case $n = 1$ [15]) and Dedecker's approach [11]. In the parametric approach the variational principle is formulated on the space $\operatorname{Imm} J^r_0(\mathbb{R}^n, E)$ under the hypothesis that the Lagrangian commute with the action of the group of parametrisations (see subsection 1.2). This leads to extra computations in order to verify at each step the invariance of objects with respect to changes of parametrisation.

Dedecker tried to reproduce the approach on jets of fibrings, but he was forced to use families of Lagrangians defined on open subsets with the property that, on intersecting subsets, the action be the same. This considerably complicated his formalism. Instead we are able to use single objects as Lagrangians because of the introduction of the pseudo-horizontal bundle and horizontalisation.

We formulate variational principles on jets of submanifolds in a way as close as possible to the case of jets of fibrings [29, 30, 39, 41, 47] (which is a close rephrasing in geometric terms of the standard variational principle). The result is very clean and provides an interpretation of the variational sequence in terms of calculus of variations.

**Definition 6.** A form $[\alpha] = h^{0,n}(\alpha) = \lambda \in \bar{\Lambda}^n_r$ is said to be an *$r$-th order generalised Lagrangian.*

Indeed, $\lambda$ depends on $(r+1)$-st derivatives in the way specified in equation (11) (*i.e.*, through hyperjacobians; see [36, 38]).

**Definition 7.** The *action* of the Lagrangian $\lambda$ on an $n$-dimensional oriented submanifold $L \subset E$ with compact closure and regular boundary is the real number

$$\mathcal{A}_L(\lambda) \overset{\text{def}}{=} \int_L (j_r L)^* \alpha. \tag{28}$$

Due to lemma 2, only the horizontal part of a form $\alpha$ contributes to the action, so that the action itself is well-defined.

*Remark* 12. We are able to introduce a distinguished Lagrangian form; it represents the whole class of its Lepage equivalents [11]. In [8, 9] instead, the authors introduce a distinguished Lepage equivalent called Hilbert–Carathéodory form. However, it appears that such a form can be introduce only if the Lagrangian form is non-vanishing. We do not need such an hypothesis.

Now, we formulate the variational problem, *i.e.*, the problem of finding extremals of the action. Let $L \subset E$ be as in the above definition. A vector field $X$ on $E$ vanishing on $\partial L$ is said a *variation field*. The submanifold $L$ is *critical* if for each variation field $X$ with flow $\phi_t$ we have

$$\frac{d}{dt}\Big|_{t=0} \int_L (J_r\phi_t \circ j_r L)^* \alpha = 0 \tag{29}$$

where $J_r\phi_t : J^r(E,n) \longrightarrow J^r(E,n)$ is the jet prolongation of $\phi_t$ (see subsection 1.1).

We shall show that, indeed, the above condition depends on the vertical part $X_V$ of $X$, and provide the Euler–Lagrange equations. First of all, we observe that $X_{rV} = \ni_{X_V}$ [7]. We have

$$\frac{d}{dt}\Big|_{t=0} \int_L (J_r\phi_t \circ j_r L)^* \alpha = \int_L (j_r L)^* \mathcal{L}_{X_r} \alpha \tag{30}$$

$$= \int_L (j_r L)^* i_{X_r} d\alpha \tag{31}$$

$$= \int_L (j_{r+1} L)^* i_{\ni_{X_V}} h^{1,n}(d\alpha) \tag{32}$$

$$= \int_L (j_{2r+1} L)^* i_{X_V} \tilde{e}_1(\lambda). \tag{33}$$

Here, $\mathcal{L}_{X_r}$ stands for Lie derivative. Equation (31) comes from Stokes'theorem and $\mathcal{L}_{X_r} = i_{X_r} d + d i_{X_r}$. Equation (32) comes from lemma 2 and the identity $h(i_{X_r} d\alpha) = i_{\ni_{X_V}} h^{1,n}(d\alpha)$, which is a direct consequence of the definition of $h^{1,n}$ (15). Finally, equation (33) comes from the identities $i_{\ni_{X_V}} \bar{d}\beta = \bar{d} i_{\ni_{X_V}} \beta$ and $(j_{r+1}L)^* \bar{d}\lambda = d(j_{r+1}L)^*\lambda$, for $\beta \in E_0^{1,q}$, implying that the value of the integral depends on the value of the $\bar{d}$-cohomology class of $h^{1,n}(d\alpha)$.

By virtue of the fundamental lemma of calculus of variations, equation (33) vanishes if and only if

$$(j_{2r+1}L)^* \tilde{e}_1(\lambda) = 0,$$

or, that is the same $\tilde{e}_1(\lambda) \circ j_{2r}L = 0$.

*Remark* 13. We obtain intrinsic Euler–Lagrange equation in our scheme, for an arbitrary number of independent variables and order of Lagrangian. To do this, we followed closely the case of jets of fibrings, hence our construction is very natural [29, 30, 39, 41, 47].

Obtaining the same in the parametric framework [8, 9] is much more involved. One of the problems is that the authors must prove at each step that the objects that they compute have the required invariance with respect to the change of parametrisation. Instead, our objects are always parametrisation-independent.

The interpretation of the variational sequence in terms of calculus of variations is now clear:

- $\bar{\Lambda}^n_r$ is the space of *Lagrangians*,
- $E_1^{1,n}$ is the space of *Euler–Lagrange type forms*,
- $E_1^{2,n}$ is the space of *Helmholtz–Sonin type forms*;
- $\tilde{e}_1$ takes a Lagrangian into its Euler–Lagrange form, the vanishing of $\tilde{e}_1$ implies that the Lagrangian is trivial (or *null*, see [3, 36]);
- $e_1$ takes an Euler–Lagrange type form into its Helmholtz-Sonin form, the vanishing of $e_1$ implies that the Euler–Lagrange type form comes from a Lagrangian.

There is no interpretation in terms of known quantities from the calculus of variations of variational forms for $p \geq 3$ and their differential for $p \geq 2$.

We think that the above variational formalism provides one of the main motivations for the $\mathcal{C}$-spectral sequence itself.

*Remark* 14. We can state what is the form of the most general null Lagrangian $\lambda \in \bar{\Lambda}^n_r$: locally, it is of the form $\bar{d}p$, where $p \in \bar{\Lambda}^{n-1}_r$. Its coordinate expression is (17), with $q = n$.

Similar statements can be made at any point of the finite order variational sequence, and answer very cleanly to questions like the ones considered in [10] (namely, the structure of null Lagrangians in the parametric formalism).

For example, a *symplectic operator* is an element $B \in E_1^{2,n}$ such that $e_1(B) = 0$ [7] (also known as *Dirac structure* in [13]). The local exactness of the finite order variational sequence ensures the local existence of a potential of $B$ in $E_1^{1,n}$ of a definite order and a definite structure of its coefficients.

*Remark* 15. Despite the fact that we obtained the above intrinsic theory, we regret that we still cannot support with interesting examples, like the ones in [8], the previous considerations, for reasons of time and space. This will be the subject of forthcoming work. Partial results have been collected in [32].

# 5 Conclusions

This work completes our research on the geometry of jets of submanifolds and their finite order $\mathcal{C}$-spectral sequence. But there is one important step further in finite order theories: the study of the constrained case, *i.e.*, to repeat the above analysis when the given space is no longer $J^r(E,n)$ but a submanifold $\mathcal{E} \subset J^r(E,n)$. This will be the subject of future research.

Moreover, we think that our research constitutes a starting point for a geometric analysis of properties of variational equations, like the geodesic equation [32], the minimal surfaces equation, etc.. Unfortunately, the development of specific examples is still incomplete; we hope to be able to produce them in a relatively short time.

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